Nonlinear plasma waves and Landau damping

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PRESENTATION

The presentation is based on the following topics:

- What is plasma?
- Mathematical models for plasma
- Solitary waves
- Dusty plasma
- Resonance particles and Landau damping

What is plasma?

Historical background:

- Greek origin ($\pi\lambda\alpha\sigma\mu\alpha$) $\Longrightarrow$ Johannes Purkinje (1787-1869)
- Irvin Langmuir in 1927 first used the term "plasma" for an ionized gas.

Plasma: A plasma is a 'quasineutral' gas of charged and neutral particles which exhibits 'collective behavior'.

(i) Quasineutrality: $\Longrightarrow$ Debye shielding

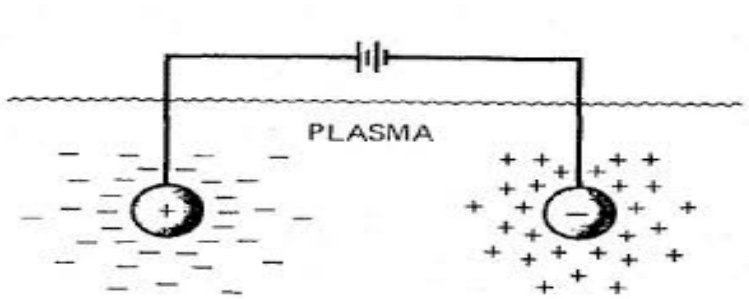


Figure: Debye shielding [Debye length is $\lambda_D = \left(\frac{\epsilon_0 K T_e}{ne^2}\right)^{1/2}$]

Solution of Poisson's equation $\epsilon_0 \frac{d^2\phi}{dx^2} = -e(n_i - n_e), \quad (Z = 1) \Longrightarrow \phi = \phi_0 exp(-|x|/\lambda_D).$

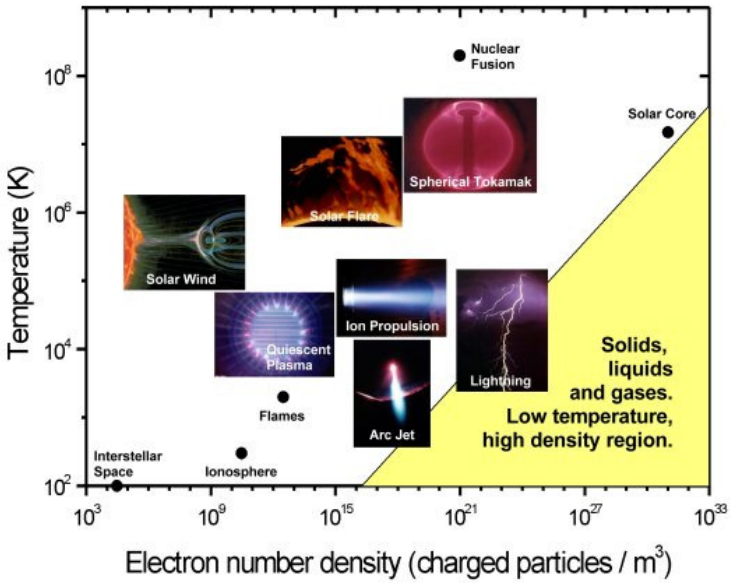
What is plasma? (continue...)

(ii) Collective behavior: An ionized gas can exhibits collective behavior in the interaction among charged particles when long-range (Coulomb) forces prevail over short-range forces.
$\Longrightarrow$ Hence, elements of plasma exert a force on one another even at large distance.

The plasma parameter and Criteria for plasmas

- It is computed that the number N_D of particles in a 'Debye sphere' $N_D = \frac{4}{3} n\pi\lambda_D^3 = 1.38 \times 10^6 T^{3/2}/n^{1/2}$.
  $\implies N_D \ggg 1$, for $\lambda_D \lll L$ where L is the characteristic length of the system.
  $\implies$ Therefore, the basic parameters are the density and the temperature.
- Therefore the first two criteria for plasmas are
  (i) $\lambda_D \lll L$,
  (ii) $N_D \ggg 1$,
  the third condition has to do with collision $\implies$ (iii) $\omega\tau > 1$

Plasma state and Some examples for plasma

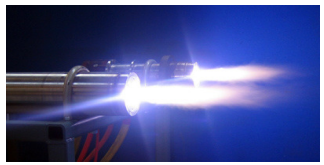




(a) Aurora Borealis



(b) Lightning



(c) Plasma torch



(d) Fluorescent lamps

Figure: Terrestrial plasmas (a,b) and Artificially produced plasmas (c,d)

Mathematical models for plasmas

Basically there are two models to model plasmas:

- Hydrodynamic model$\Longrightarrow$

$$\frac{\partial n_e}{\partial t} + \frac{\partial (n_e u_e)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} = -\frac{q_e}{m_e}\frac{\partial \phi}{\partial x}, \tag{2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e(n_e - n_{i0}). \tag{3}$$

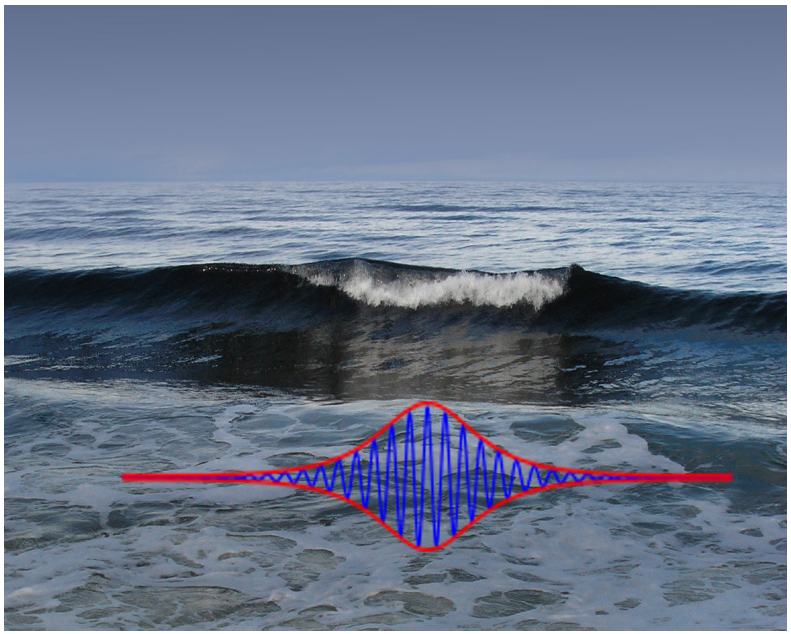
- Kinetic model$\Longrightarrow$

$$\frac{\partial f_e}{\partial t} + v_e \frac{\partial f_e}{\partial x} - \frac{q_e}{m_e}\frac{\partial \phi}{\partial x}\frac{\partial f_e}{\partial v_e} = 0, \tag{4}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e\left(\int_{-\infty}^{\infty} f_e dv - n_{i0}\right). \tag{5}$$

What is solitary wave?

- Solitary waves are hump/dip shaped nonlinear waves of permanent profile due to the balance between nonlinearity and dispersion (without dissipation). However, when dissipation is important, shock waves are formed.



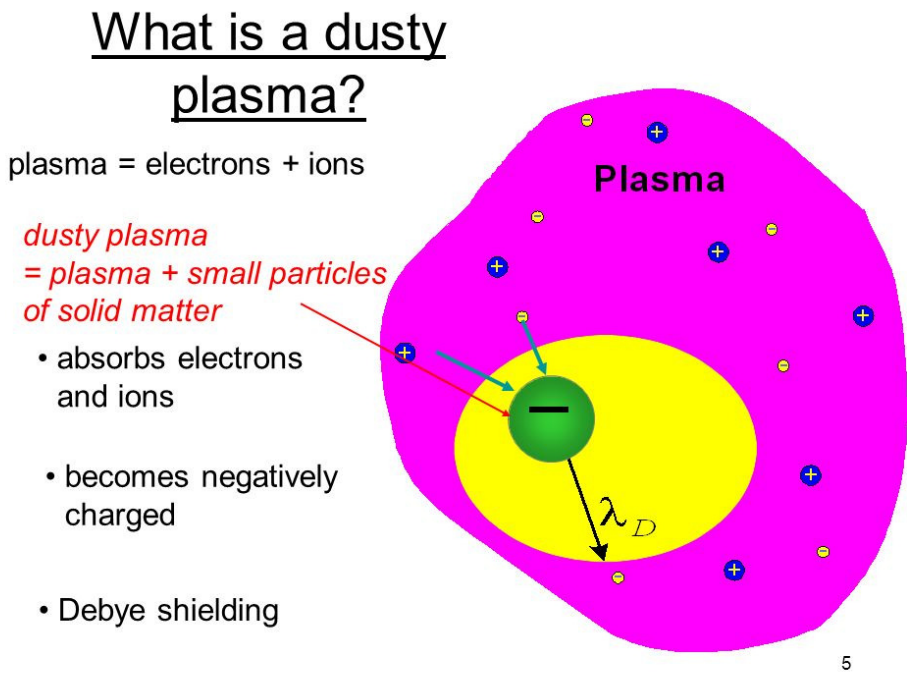
(a)



(b)

Figure: Solitary waves

What is Dusty plasma?



- Dusty plasma contains electrons, ions and neutrals particles with dust having size μm - mm, not constant in general. The mass of the dust grain billions times heavier than ions, not constant in general. And the charge is not neutral rather $+$ or - depending on charging processes [Verheest (2000); Shukla and Mamun (2002)].

Dust acoustic solitary waves

There are two normal modes (waves) which are experimentally observed in dusty plasmas:

1. Static Dust (DIA Waves): $\omega/k = \sqrt{(n_{i0}/n_{e0})(T_e/m_i)}$
The restoring force comes from the electron thermal pressure and the inertia is provided by the ion mass. The dust charge maintains only the equilibrium charge neutrality [theoretically predicted by Shukla and Silin (1992) and experimentally observed by D'Angelo et al (1996)].

2. Mobile Dust (DA Waves): $\omega/k \approx \sqrt{(Z_d n_{d0}/n_{i0})(Z_d T_i/m_d)}$
The restoring force mainly comes from ion thermal pressure and the inertia is provided by dust mass. This types are in general very low frequency modes and can be visible with naked eyes [theoretically predicted by Rao et al (1990) and experimentally observed by Barkan et al (1995)].

Resonance particles and Landau damping

- In plasmas the particles may have the velocity which approaches the wave phase velocity or group velocity. Such particles can be trapped and/or free. This type of particles are called resonance particles.

- The waves in plasmas can undergo collisionless damping when they resonantly interact with resonance particles. Such kind of wave damping known as Landau damping [J. Phys. (USSR) 10, 25 (1946)].

Importance of Landau damping

In plasma oscillations

- Individual particles interact with the field produced by all particles.
- Changes behaviour of the particles.
- They can change the field producing the forces.
  Thus, the particles may have different velocities.
- So, a self-consistent treatment is required. And if we allow ω to be complex ($\omega = \omega_r + i\omega_i$) then,

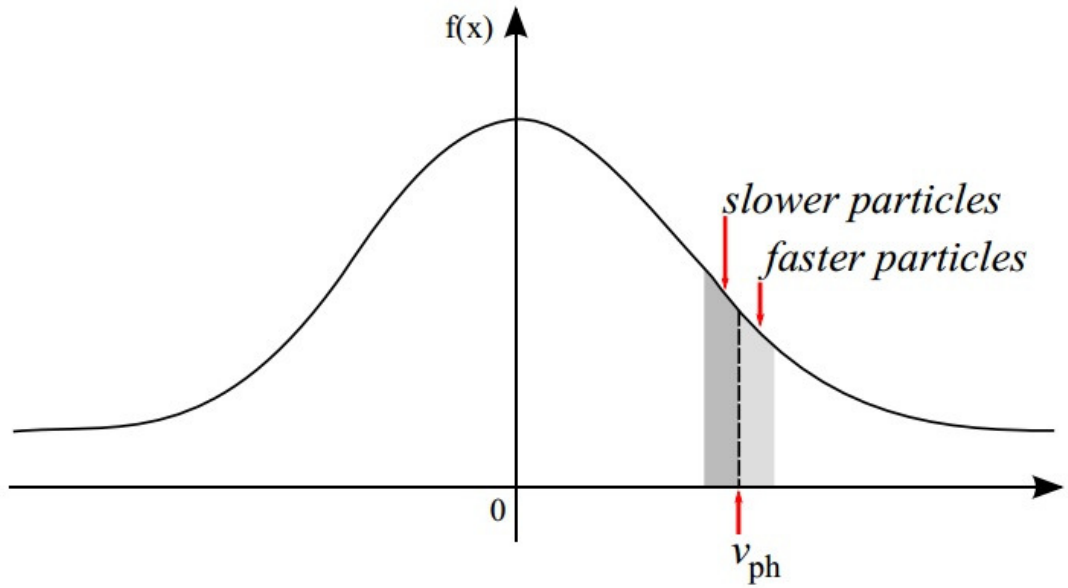
$$E(x,t) = E_0 e^{i(kx-\omega t)} \rightarrow E(x,t) = E_0 e^{i(kx-\omega_r t)} e^{\omega_i t}$$

and we can have a damped oscillation for $\omega_i < 0$

Importance of Landau damping (interaction with mode)

For different velocity particles:

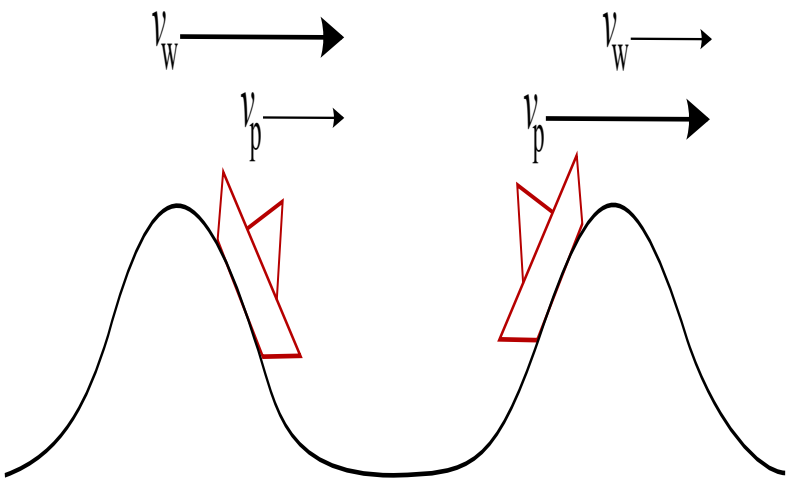
- if more particles are moving slower than the wave then the net absorption of energy will be from the wave and hence the wave is damped.
- if more particles are moving faster than the wave then the net absorption of energy will be by the wave and the wave is anti-damped. Therefore, the slope of the particle distribution at the wave velocity is always important.



Importance of Landau damping (interaction with mode): real-life problem

In real life problem, if a surfer trying to catch the wave, moving in the same direction, then

- if surfer faster than wave, wave gains energy from the surfer.
- if surfer slower than wave, wave loses energy to the surfer.



(a) $v \leqslant v_{ph}$ ⇒particle gains energy
$v \geqslant v_{ph}$ ⇒wave gains energy



(b) Real life problem

Importance of Landau damping (interaction with mode): real-life problem

But, it does not always work like that. There may exits two extreme cases:

- Surfer very fast: *jumps* across the wave crests, little interaction with the wave.
- Surfer not moving: *oscillates* up and down with the waves.

Thus, wave velocity and surfer velocity must be similar otherwise surfer is *trapped* by the wave.

- Basic equations (Dimensionless form):

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{v}_j) = 0, \tag{6}$$

$$\frac{d\mathbf{v}_j}{dt} = -\zeta_j \left(\frac{m_n}{m_j} \nabla \phi - \mathbf{v}_j \times \omega_{cj} \hat{z} \right) - \frac{5}{3} \frac{m_n T_j}{m_j T_p} n_p^{-1/3} \nabla n_j, \tag{7}$$

$$\nabla^2 \phi = n_n - \mu n_p - \delta, \tag{8}$$

together with the charge neutrality condition given by

$$\mu + \delta = 1, \tag{9}$$

where $\zeta_p = 1$, $\zeta_n = -1$, $\omega_{cj} = eB_0/cm_j\omega_{pn}$ is the cyclotron frequency for the j-th species of ions, normalized by the negative-ion plasma frequency, $\mu = n_{p0}/n_{n0}$, and $\delta = z_d n_{d0}/n_{n0}$.

KdV equation:

$$\frac{\partial \Phi}{\partial \tau} + A\Phi\frac{\partial \Phi}{\partial \xi} + B\frac{\partial^3 \Phi}{\partial \xi^3} = 0, \tag{10}$$

where $\Phi \equiv \phi^{(1)} \lesssim 1$, such that $\epsilon\phi^{(1)} < 1$. The coefficients of nonlinearity and dispersion are, respectively, given by

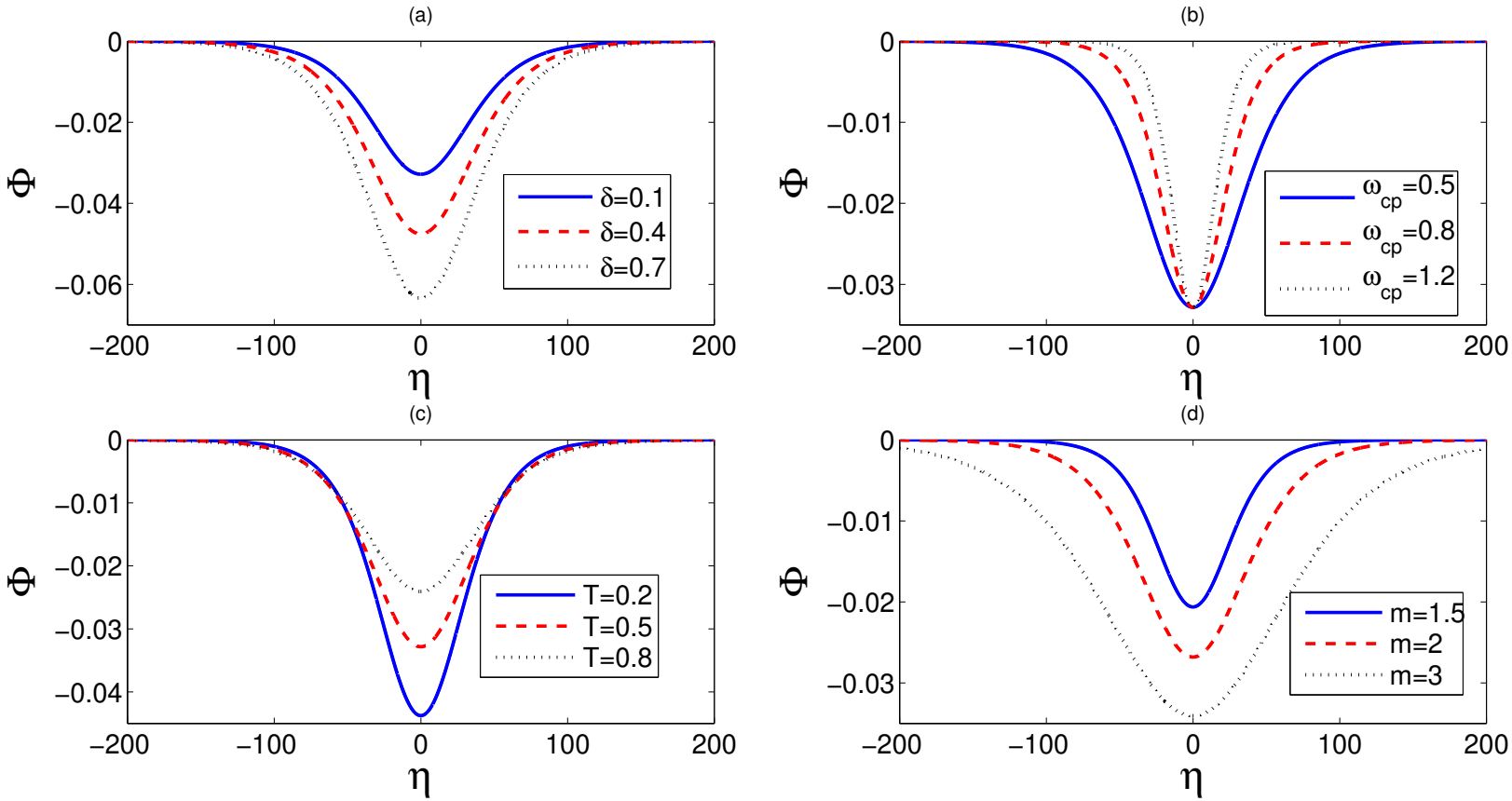
$$\begin{aligned} A \quad &= \frac{1}{2\sqrt{15}}\frac{l_z}{(m-T)}\sqrt{\frac{m}{(1+\mu m)(1+\mu T)}} \\ &\times \left[\left(1+\mu^2 T\right)(1+\mu m) - 9\left(1+\mu^2 m\right)(1+\mu T)\right], \end{aligned} \tag{11}$$

$$\begin{aligned} B \quad &= \frac{5\sqrt{5}}{6\sqrt{3}} l_z \sqrt{m}\frac{(1+\mu T)^{3/2}}{(1+\mu m)^{5/2}} \\ &\times \left[\frac{\mu(m-T)^2}{(1+\mu T)^2} + \frac{(1-l_z^2)(1+m^3\mu)}{\omega_{cp}\omega_{cn}}\right]. \end{aligned} \tag{12}$$

- Solution of Eq. (10):

$$\Phi = \Phi_m \mathrm{sech}^2(\eta/w), \tag{13}$$

where $\Phi_m = 3U_0/A$ is the finite amplitude and $w = \sqrt{4B/U_0}$ is the finite width of the soliton with $B > 0$.



- Critical value m_c of mass ratio m is:

$$m_c = \frac{8 + \mu T(9 - \mu)}{\mu\left[1 - \mu(9 + 8\mu T)\right]}. \tag{14}$$

- For $m \sim m_c$ the KdV equation (10) fails to describe the evolution of DIA waves.
- One has go for higher-order corrections to derive a modified KdV equation.
- Derivation of mKdV equation: Expansions

$$\begin{aligned}
n_j &= 1 + \epsilon^{1/2} n_j^{(1)} + \epsilon n_j^{(2)} + \epsilon^{3/2} n_j^{(3)} + \cdots, \\
v_{jx,y} &= \epsilon v_{jx,y}^{(1)} + \epsilon^{3/2} v_{jx,y}^{(2)} + \epsilon^2 v_{jx,y}^{(3)} + \cdots, \\
v_{jz} &= \epsilon^{1/2} v_{jz}^{(1)} + \epsilon v_{jz}^{(2)} + \epsilon^{3/2} v_{jz}^{(3)} + \cdots, \\
\phi &= \epsilon^{1/2} \phi^{(1)} + \epsilon \phi^{(2)} + \epsilon^{3/2} \phi^{(3)} + \cdots.
\end{aligned} \tag{15}$$

- mKdV Equation:

$$\frac{\partial \Phi}{\partial \tau} + A' \Phi^2 \frac{\partial \Phi}{\partial \xi} + B \frac{\partial^3 \Phi}{\partial \xi^3} = 0, \tag{16}$$

where $\Phi \equiv \phi^{(1)}$ and

$$\begin{aligned} A' = \quad & \frac{1}{10} \frac{l_z}{\mu^3} \left[\frac{3}{5} m(1+\mu T) \right]^{1/2} \frac{(1+\mu m)^{3/2}}{(m-T)^3} \\ & \times \left[\frac{45}{2} \left(\frac{1+\mu T}{1+\mu m} \right) \left(\mu^4 m^2 - 1 \right) + 3 \left(\mu^4 m T - 1 \right) \right. \\ & + \frac{1}{2} \frac{\left(\mu^4 T^2 - 1 \right) (1+\mu m)}{1+\mu T} \\ & \left. + \frac{2}{3} \frac{\mu (1+\mu^3 T)(m-T)}{1+\mu T} \right]. \end{aligned} \tag{17}$$

- Solution of Eq. (16):

$$\Phi = \Psi \, \mathrm{sech}\,(\eta/W)\,, \tag{18}$$

where $\Psi = 6U_0/A'$ is the amplitude and $W = \sqrt{B/U_0}$ is the width of the mKdV soliton with $B > 0$.

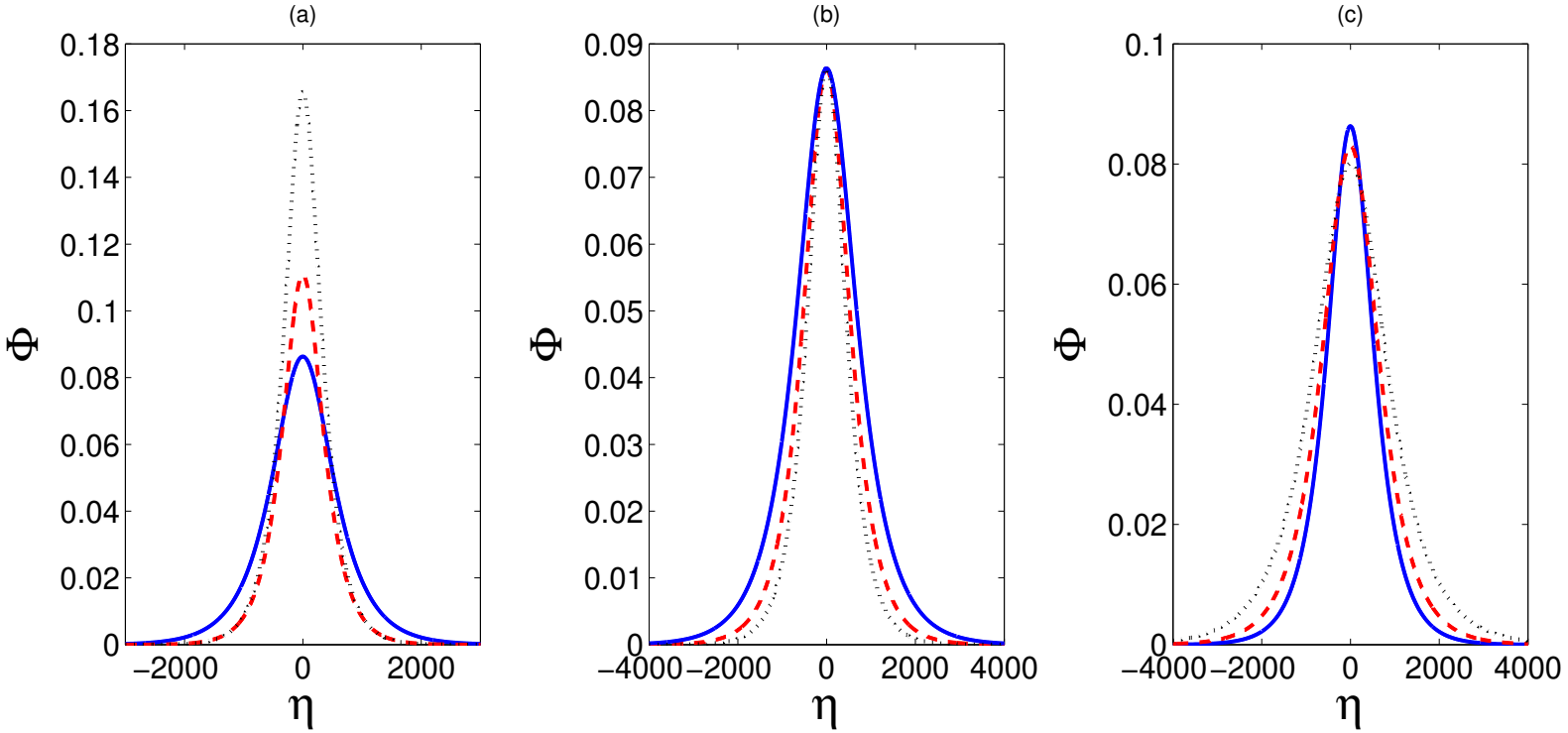


Figure: Effects of (a) δ (*positively charged dusts*), (b) ω_{cp} (*external magnetic field*) and (c) T (*temperature ratio*).

2. Landau damping effects on dust-acoustic solitary waves in a dusty negative-ion plasma [Phys. Plasmas **21**, 073708 (2014)]

- Basic equations (Dimensionless form):

$$\frac{\partial n_d}{\partial t} + \frac{\partial (n_d u_d)}{\partial x} = 0, \tag{19}$$

$$\frac{\partial u_d}{\partial t} + u_d \frac{\partial u_d}{\partial x} = -\zeta \frac{\partial \phi}{\partial x}, \tag{20}$$

$$\delta \frac{\partial f_j}{\partial t} + v \frac{\partial f_j}{\partial x} - \zeta_j \frac{m_p}{m_j} \frac{\partial \phi}{\partial x} \frac{\partial f_j}{\partial v} = 0, \tag{21}$$

$$\frac{\lambda_D^2}{L^2} \frac{\partial^2 \phi}{\partial x^2} = \mu_{nd} n_n - \mu_{pd} n_p - \zeta n_d, \tag{22}$$

$$n_j = \sqrt{\frac{m_j}{m_p} \frac{T_p}{T_j}} \int_{-\infty}^{\infty} f_j dv, \tag{23}$$

together with the charge neutrality condition $\mu_{pd} + \zeta = \mu_{nd}$, where in Eq. (21), $\zeta_j = \pm 1$ for positive ($j = p$) and negative ($j = n$) ions and $\delta = \sqrt{z_d m_p / m_d}$. Also, in Eq. (22), $\mu_{jd} = n_{j0}/z_d n_{d0}$ for $j = p$ and n.

Expansions:

$$\begin{aligned} n_d &= 1 + \alpha_2 \epsilon n_d^{(1)} + \alpha_2^2 \epsilon^2 n_d^{(2)} + \cdots , \\ u_d &= \alpha_2 \epsilon u_d^{(1)} + \alpha_2^2 \epsilon^2 u_d^{(2)} + \cdots , \\ \phi &= \alpha_2 \epsilon \phi^{(1)} + \alpha_2^2 \epsilon^2 \phi^{(2)} + \cdots , \\ n_j &= 1 + \alpha_2 \epsilon n_j^{(1)} + \alpha_2^2 \epsilon^2 n_j^{(2)} + \cdots , \\ f_j &= f_j^{(0)} + \alpha_2 \epsilon f_j^{(1)} + \alpha_2 \epsilon^2 f_j^{(2)} + \cdots , \end{aligned} \tag{24}$$

Unperturbed velocity distribution for positive and negative ions (Maxwellian form):

$$f_j^{(0)} = \sqrt{1/2\pi} \exp\left[\left(m_j T_p / m_p T_j \right) \left(-v^2/2 \right) \right] . \tag{25}$$

KdV equation with Landau damping:

$$\frac{\partial n}{\partial \tau} + aP \int_{-\infty}^{\infty} \frac{\partial n}{\partial \xi'} \frac{d\xi'}{\xi - \xi'} + bn \frac{\partial n}{\partial \xi} + c \frac{\partial^3 n}{\partial \xi^3} = 0, \tag{26}$$

where $n \equiv n_d^{(1)}$ and the coefficients of the Landau damping, nonlinear and dispersive terms are

$$a = \frac{\alpha_1}{\sqrt{8\pi}} \frac{\sigma^{-1/2}}{\sigma_1^2} \left[\zeta \sqrt{m} + (\sqrt{m} + \sigma^{-3/2}) \mu_{pd} \right], \tag{27}$$

$$b = \frac{\alpha_2}{2\sqrt{\sigma \sigma_1}} \left[3 - \zeta \frac{\sigma_2}{\sigma_1^2} \right], \tag{28}$$

$$c = \frac{\alpha_3}{2} (\sigma \sigma_1)^{-3/2}, \tag{29}$$

with $\sigma_1 = \zeta + \left(1 + \sigma^{-1}\right) \mu_{pd}$, $\sigma_2 = \zeta + \left(1 - \sigma^{-2}\right) \mu_{pd}$ and $\zeta = \pm 1$ for positively and negatively charged dusts.

The soliton solution of Eq. (26) with the Landau damping is given by

$$n = N_0 \left(1 + \frac{\tau}{\tau_0}\right)^{-2} \operatorname{sech}^2 \left[\left(\xi - \frac{b}{3}\int_0^\tau N d\tau\right)/W\right] + O(a), \tag{30}$$

where $N = N_0$ at $\tau = 0$ and τ_0 and $N(a, \tau)$ are given by

$$\tau_0^{-1} = \frac{a}{4}\sqrt{\frac{bN_0}{3c}} \mathrm{P} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\operatorname{sech}^2 z}{z - z'} \frac{\partial}{\partial z'} \left(\operatorname{sech}^2 z'\right) dz dz', \tag{31}$$

and

$$N(a, \tau) = N_0 \left(1 + \frac{\tau}{\tau_0}\right)^{-2}. \tag{32}$$

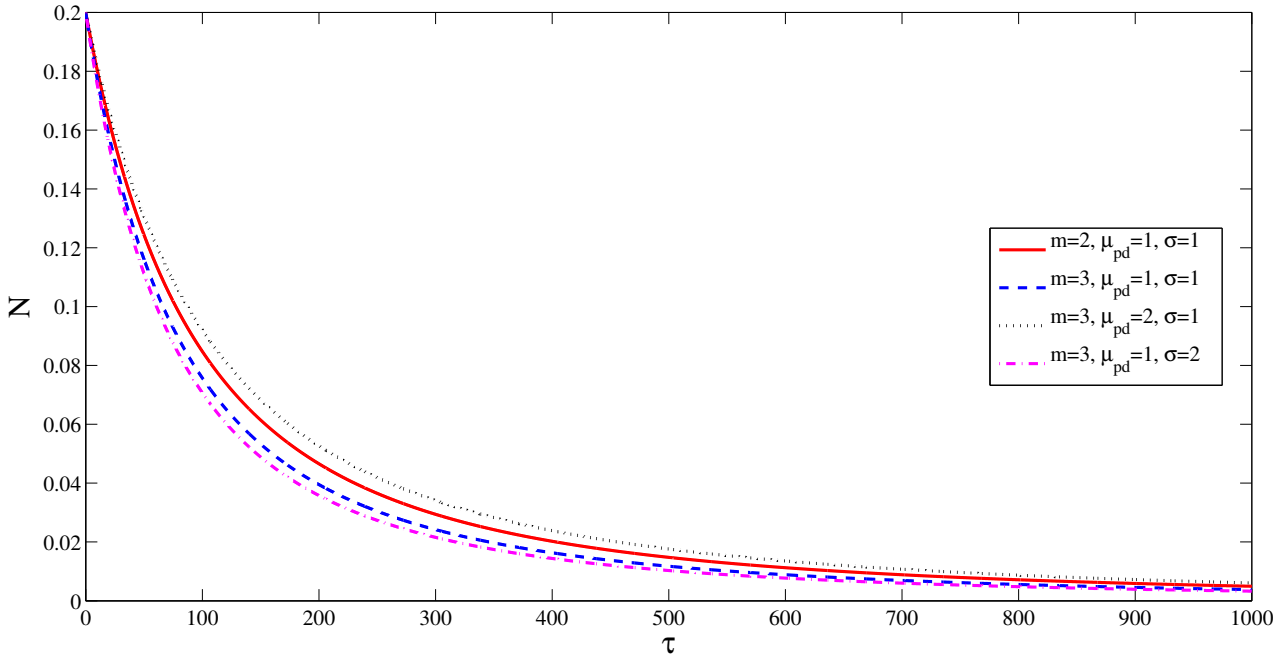
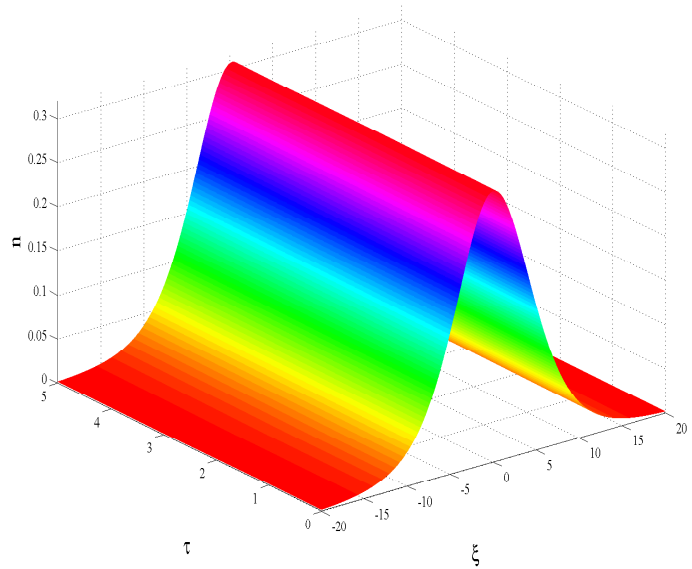
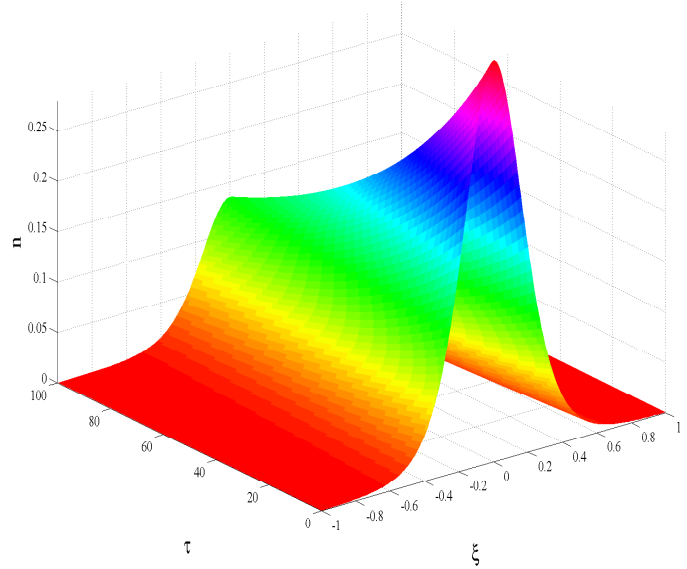


Figure: The decay of the solitary wave amplitude [Eq. (32)] with time is shown as a result of the Landau damping effect for different values of the parameters m, μ_{pd} and σ. The qualitative behaviors remain the same for plasmas with negatively charged dusts.



(a) Soliton without Landau damping



(b) Soliton with Landau damping

Figure: Typical forms of the soliton solutions.

- Basic equations (Dimensionless form):

$$\partial_t n_d + \partial_x (n_d u_d) = 0, \tag{33}$$

$$\partial_t u_d + u_d \partial_x u_d = \partial_x \phi, \tag{34}$$

$$\delta \partial_t f_j + v \partial_x f_j - (m_h/m_j) \partial_x \phi \partial_v f_j = 0, \tag{35}$$

$$\left(\lambda_D^2/L^2\right) \partial_x^2 \phi = -(\mu_{cd} n_c + \mu hd\, n_h + n_d), \tag{36}$$

$$n_j = \sqrt{m_j T_h / m_h T_j} \int_{-\infty}^{\infty} f_j dv, \tag{37}$$

where $\delta = \sqrt{z_d m_h / m_d}$ and $\mu_{jd} = n_{j0}/z_d n_{d0}$ is the density ratio for $j = c,\ h$ together with the charge neutrality condition

$$\mu_{cd} + \mu_{hd} = 1, \text{ or } \mu_{hd} = \mu/(1+\mu), \tag{38}$$

where $\mu = n_{h0}/n_{c0}$ is the ratio of hot to cold unperturbed ion densities.

- Derivation of KdV equation: Stretching:

$$\xi = \epsilon^{1/2}(x - Mt),\ \tau = \epsilon^{3/2}t, \tag{39}$$

Expansions:

$$\begin{aligned} n_d &= 1 + \alpha_2\epsilon n_d^{(1)} + \alpha_2^2\epsilon^2 n_d^{(2)} + \cdots, \\ u_d &= \alpha_2\epsilon u_d^{(1)} + \alpha_2^2\epsilon^2 u_d^{(2)} + \cdots, \\ \phi &= \alpha_2\epsilon\phi^{(1)} + \alpha_2^2\epsilon^2\phi^{(2)} + \cdots, \\ n_j &= 1 + \alpha_2\epsilon n_j^{(1)} + \alpha_2^2\epsilon^2 n_j^{(2)} + \cdots, \\ f_j &= f_j^{(0)} + \alpha_2\epsilon f_j^{(1)} + \alpha_2^2\epsilon^2 f_j^{(2)} + \cdots, \end{aligned} \tag{40}$$

where the equilibrium distribution function $f_j^{(0)}$, $j = c,\ h$, are assumed to be the Maxwellian

$$f_j^{(0)} = \sqrt{1/2\pi}\exp\left[(m_j T_e/m_e T_j)(-v^2/2)\right]. \tag{41}$$

- KdV equation:

$$\frac{\partial n}{\partial \tau} + aP \int_{-\infty}^{\infty} \frac{\partial n}{\partial \xi'} \frac{d\xi'}{\xi - \xi'} + bn \frac{\partial n}{\partial \xi} + c \frac{\partial^3 n}{\partial \xi^3} = 0, \tag{42}$$

where the coefficients of the Landau damping (a), the nonlinear (b) and the dispersive term (c), respectively, are given by

$$a = \left(\alpha_1 / \sqrt{8\pi\sigma}\sigma_1^2\right) \left[\sqrt{m} - (\sqrt{m} - \sigma^{-3/2})\mu_{hd}\right], \tag{43}$$

$$b = (\alpha_2 / 2\sqrt{\sigma\sigma_1}) \left(3 - \sigma_2/\sigma_1^2\right). \tag{44}$$

$$c = (\alpha_3/2)(\sigma\sigma_1)^{-3/2}, \tag{45}$$

with $\sigma_1 = 1 + (\sigma^{-1} - 1)\mu_{hd} \equiv \mu_{cd}(\sigma + \mu)/\sigma$,
$\sigma_2 = 1 + (\sigma^{-2} - 1)\mu_{hd} \equiv \mu_{cd}(\sigma^2 + \mu)/\sigma^2$ and
$\sigma_3 = 1 + (\sigma^{-3} - 1)\mu_{hd} \equiv \mu_{cd}(\sigma^3 + \mu)/\sigma^3$.

- Derivation of Gardner and mKdV equations: The nonlinear coefficient b (Eq. (44)) of the KdV equation (42) vanishes for $R_1 \equiv 3 - \sigma_2/\sigma_1^2 \equiv 3 - (1+\mu)(\sigma^2+\mu)/(\sigma+\mu)^2 = 0$ along a curve in the $\sigma - \mu \Longrightarrow$ Gardner (KdV-mKdV) equation with Landau damping:

$$\frac{\partial n}{\partial \tau} + aP\int_{-\infty}^{\infty} \frac{\partial n}{\partial \xi'}\frac{d\xi'}{\xi - \xi'} + b_1 n\frac{\partial n}{\partial \xi} + b_2 n^2 \frac{\partial n}{\partial \xi} + c\frac{\partial^3 n}{\partial \xi^3} = 0, \tag{46}$$

where the coefficients a and c, corresponding to the Landau damping and dispersive effects, are the same as for the KdV equation (42). The nonlinear coefficients b_1 and b_2 are, however, different and given by

$$b_1 = \left(3 - \sigma_2/\sigma_1^2\right)/2\sqrt{\sigma\sigma_1}, \tag{47}$$

$$b_2 = (15/4)\left(\alpha_2/(\sigma\sigma_1)^{1/2}\right)\left[1 - \left(\sigma_3/15\sigma_1^3\right)\right]. \tag{48}$$

Here, the expressions for σ_1 and σ_2 are given before and $\sigma_3 = (\sigma^3+\mu)/\sigma^3(1+\mu)$.

- Solution of Gardner equation Eq. (46): Rescaling Eq. (46) and considering a travelling wave solution as

$$n(\xi,\tau') = 6\bar{U}\left[b_1' \pm \bar{N}\cosh\left((\xi - \bar{U}\tau')/\bar{W}\right)\right]^{-1}, \tag{49}$$

where $\bar{W} = \sqrt{6b_2'/(\bar{N}^2 - b_1'^2)}$ is the constant width and $\bar{U} = (\bar{N}^2 - b_1'^2)/6b_2'$ is the constant phase speed (normalized by c_s) of the DASWs, we have the solution of Gardner Eq. (46) as

$$\bar{N}(a,\tau) = (3b_2/c)^{1/4}\bar{N}_0\,(\lambda_3/\lambda_4)\,, \tag{50}$$

where $\bar{N} = \bar{N}_0$ at $\tau = 0$ and

$$\lambda_3 = 3^{5/4}(b_2/c)^{1/4}\bar{N}_0 + \zeta\left[(3b_2/c)^{1/2} \times(3\bar{N}_0 \mp \pi b_1/c)^2 \pm 2\sqrt{2}\pi b_1 A' a\bar{N}_0^2\tau/c\right]^{1/2}, \tag{51}$$

$$\lambda_4 = \sqrt{3b_2/c}\,(6\bar{N}_0 \mp 2\pi b_1/c) - \sqrt{2}A' a\bar{N}_0^2\tau, \tag{52}$$

$$A' = \mathrm{P}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{sech}z\frac{\partial}{\partial z'}(\mathrm{sech}z')\frac{dz'}{z-z'}dz. \tag{53}$$

- Basic equations (Dimensionless form):

$$\frac{\partial f_e}{\partial t} + v\frac{\partial f_e}{\partial x} + \frac{1}{m}\frac{\partial \phi}{\partial x}\frac{\partial f_e}{\partial v} - \frac{H^2}{24m^2}\frac{\partial^3 \phi}{\partial x^3}\frac{\partial^3 f_e}{\partial v^3} = 0, \tag{54}$$

$$\frac{\partial f_i}{\partial t} + v\frac{\partial f_i}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial f_i}{\partial v} = 0, \tag{55}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -\sum_{\alpha=e,i} \theta_\alpha \int f_\alpha dv, \tag{56}$$

where $m = m_e/m_i$ is the electron to ion mass ratio, $H = \hbar\omega_{pe}/k_B T_e$ is the dimensionless quantum parameter denoting the ratio of the electron plasmon energy to the thermal energy and $\theta_\alpha = \mp 1$ for electrons ($\alpha = e$) and ions ($\alpha = i$).

- Derivation of KdV equation: Perturbation:

$$\begin{aligned}\phi =&\epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \cdots,\\ f_\alpha =&f_\alpha^{(0)} + \epsilon f_\alpha^{(1)} + \epsilon^2 f_\alpha^{(2)} + \cdots,\end{aligned} \tag{57}$$

$f_\alpha^{(0)}$, for $\alpha = e, i$, are assumed to be the Maxwellian given by

$$f_\alpha^{(0)} = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{m_\alpha T_e}{m_i T_\alpha}}\exp\left(-\frac{m_\alpha T_e}{m_i T_\alpha}\frac{v^2}{2}\right). \tag{58}$$

- multi-scale Fourier-Laplace transforms for $f_\alpha - f_\alpha^{(0)}$ and ϕ

$$\xi = \epsilon^{1/2}x,\ \sigma = \epsilon^{1/2}t,\ s = \epsilon^{3/2}x, \tag{59}$$

$$\begin{aligned}f_\alpha^{(n)}(v,\xi,\sigma,s) =&\frac{i}{(2\pi)^2}\int_{-\infty}^{\infty}dk\int_W d\omega \hat{f}_\alpha^{(n)}(v,k,\omega,s)\\ &\times(\omega - k\lambda)^{-1}\exp[i(k\xi - \omega\sigma)],\\ \phi^{(n)}(\xi,\sigma,s) =&\frac{i}{(2\pi)^2}\int_{-\infty}^{\infty}dk\int_W d\omega \hat{\phi}^{(n)}(k,\omega,s)\\ &\times(\omega - k\lambda)^{-1}\exp[i(k\xi - \omega\sigma)].\end{aligned} \tag{60}$$

KdV equation for IAWs:

$$\frac{\partial \phi}{\partial s} + \delta \phi \frac{\partial \phi}{\partial \zeta} + \beta \frac{\partial^3 \phi}{\partial \zeta^3} + \gamma \, \mathrm{P} \int_{-\infty}^{\infty} (\zeta - \zeta')^{-1} \frac{\partial \phi(\zeta')}{\partial \zeta'} d\zeta' = 0, \tag{61}$$

where $\phi \equiv \phi^{(1)}$ and the coefficients are given by $\delta = b/a$, $\beta = 1/a$ and $\gamma = c/a$. If the equilibrium distribution of particles be the Maxwellian [Eq. (58)], the coefficients a, b and c are

$$a = 2\lambda^{-2}\left(1 + 6\lambda^{-2}T^{-1}\right) + \lambda H^2 k, \tag{62}$$

$$b = 3\lambda^{-4} + 30\,T^{-1}\lambda^{-6} - 1, \tag{63}$$

$$c = \epsilon^{-1}\frac{\lambda}{\sqrt{2\pi}}\left[m^{1/2} + T^{3/2}\exp\left(-\frac{T\lambda^2}{2}\right)\right]. \tag{64}$$

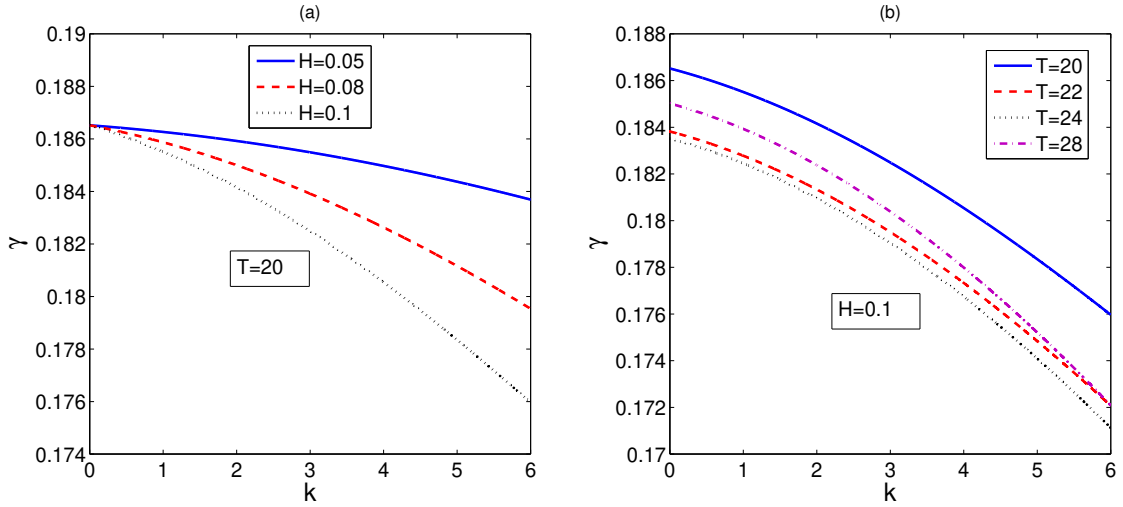
Solitary wave solution:

$$\phi = \Psi \operatorname{sech}^2 \left[\left(\zeta - \frac{\delta}{3} \int_0^s \Phi ds \right) / W \right] + o(\gamma), \tag{65}$$

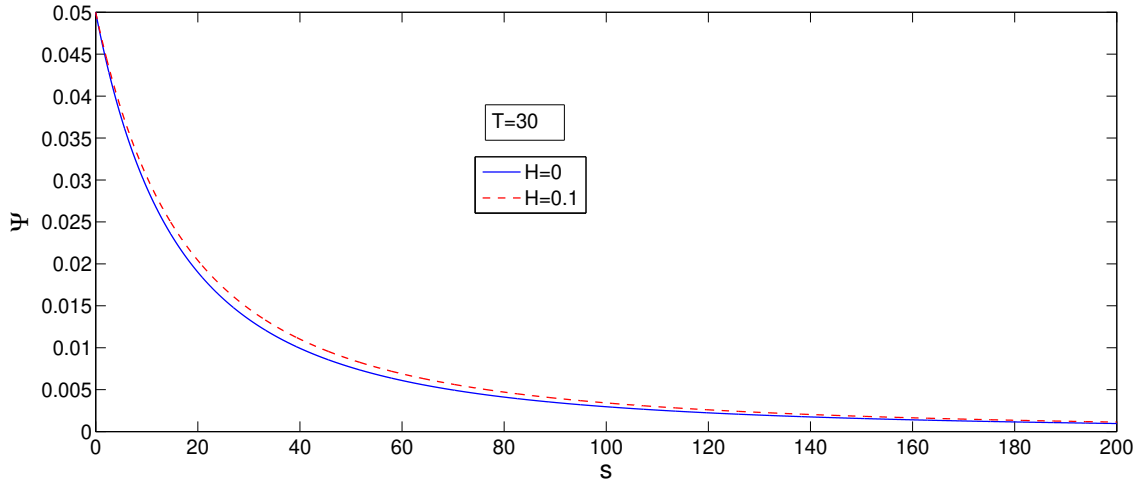
where $\Psi = \Phi_0 \left(1 + s/s_0\right)^{-2}$ is the amplitude of the solitary wave solution of the modified KdV equation (61), and $\Phi = 3U_0/\delta$ is the corresponding amplitude, $W = \left(12\beta/\Phi\delta\right)^{1/2} \equiv \sqrt{4\beta/U_0}$ is the width and $U_0 = \Phi\delta/3$ is the constant phase speed (normalized by c_s) of the solitary wave solution of the KdV equation in absence of the Landau damping (i.e., when γ or $c = 0$). Also, $\Phi = \Phi_0$ at $s = 0$ and s_0 is given by

$$s_0^{-1} = \frac{\gamma}{4} \sqrt{\frac{\delta\Phi_0}{3\beta}} \mathrm{P} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\operatorname{sech}^2 z}{z - z'} \frac{\partial}{\partial z'} \left(\operatorname{sech}^2 z' \right) dz dz'. \tag{66}$$

- Landau damping rate:



- Decay of wave amplitude Ψ:



Acknowledgement

